

# THE SUPPORTS OF HIGHER BIFURCATION CURRENTS

ROMAIN DUJARDIN

**ABSTRACT.** Let  $(f_\lambda)_{\lambda \in \Lambda}$  be a holomorphic family of rational mappings of degree  $d$  on  $\mathbb{P}^1(\mathbb{C})$ , with  $k$  marked critical points  $c_1, \dots, c_k$ . To this data is associated a closed positive current  $T_1 \wedge \dots \wedge T_k$  of bidegree  $(k, k)$  on  $\Lambda$ , aiming to describe the simultaneous bifurcations of the marked critical points. In this note we show that the support of this current is accumulated by parameters at which  $c_1, \dots, c_k$  eventually fall on repelling cycles. Together with results of Buff, Epstein and Gauthier, this leads to a complete characterization of  $\text{Supp}(T_1 \wedge \dots \wedge T_k)$ .

## INTRODUCTION

Let  $(f_\lambda)_{\lambda \in \Lambda}$  be a holomorphic family of rational mappings of degree  $d$  on  $\mathbb{P}^1(\mathbb{C})$ , parameterized by a complex manifold  $\Lambda$ . Let  $c = (c(\lambda))_{\lambda \in \Lambda}$  be a marked (i.e. holomorphically moving) critical point. It is an immediate consequence of Montel's theorem that if  $c$  bifurcates at  $\lambda_0$ , then there exist nearby parameters where the orbit of  $c$  falls eventually on a repelling cycle. Now if  $k > 1$  critical points  $c_1, \dots, c_k$  are marked, and simultaneously bifurcate at  $\lambda_0$ , it is natural to wonder whether they can be perturbed to be made simultaneously preperiodic. An example due to Douady shows that this is impossible in general (see [DF, Example 6.13]).

It appears that the right language to deal with this type of questions is that of bifurcation currents, which we briefly review now. The reader is referred to [Du3, Be] for a thorough presentation. To a marked critical point  $c$ , following [DF], one can associate a bifurcation current  $T_c$  on  $\Lambda$ , whose support coincides with the activity (bifurcation) locus of  $c$ . When  $k$  critical points  $c_1, \dots, c_k$  are marked, the wedge product  $T_1 \wedge \dots \wedge T_k$  is well defined and its support is contained (but not equal to) the locus where the  $c_i$  are simultaneously active.

Assume that for  $\lambda_0 \in \Lambda$ , the marked critical points  $c_j$  eventually land on repelling periodic points  $p_j$ , that is  $f_{\lambda_0}^{n_j}(c_j(\lambda_0)) = p_j(\lambda_0)$ . We let the mappings  $C$  and  $P : \Lambda \rightarrow (\mathbb{P}^1)^k$  be respectively defined by  $C : \lambda \mapsto (f_\lambda^{n_1}(c_1(\lambda)), \dots, f_\lambda^{n_k}(c_k(\lambda)))$  and  $P : \lambda \mapsto (p_1(\lambda), \dots, p_k(\lambda))$ , where  $p_j(\lambda)$  is the natural continuation of  $p_j(\lambda_0)$  as a periodic point. We say that the  $c_j$  *fall transversely* onto the periodic points  $p_j$  if the graphs of the mappings  $P$  and  $C$  are transverse at  $\lambda_0$ .

Our main result is the following.

**Theorem 0.1.** *Let  $(f_\lambda)_{\lambda \in \Lambda}$  be a holomorphic family of rational maps of degree  $d \geq 2$  on  $\mathbb{P}^1$ . Assume that  $c_1, \dots, c_k$  are marked critical points and let  $T_1, \dots, T_k$  be the respective bifurcation currents.*

*Then every parameter in  $\text{Supp}(T_1 \wedge \dots \wedge T_k)$  is accumulated by parameters  $\lambda$  for which  $c_1(\lambda), \dots, c_k(\lambda)$  fall transversely onto repelling cycles.*

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Conversely, Buff and Epstein [BE] showed that if at  $\lambda_0$ ,  $c_1(\lambda_0), \dots, c_k(\lambda_0)$  fall transversely onto repelling cycles, then  $\lambda_0 \in \text{Supp}(T_1 \wedge \dots \wedge T_k)$ . Altogether this leads to the following precise characterization of  $\text{Supp}(T_1 \wedge \dots \wedge T_k)$ .

**Corollary 0.2.** *Under the assumptions of Theorem 0.1,  $\text{Supp}(T_1 \wedge \dots \wedge T_k)$  is the closure of the set of parameters  $\lambda$  for which  $c_1(\lambda), \dots, c_k(\lambda)$  fall transversely onto repelling cycles.*

By using a result of Gauthier [Ga] we may replace the word “transversely” by “properly” in the above corollary (see §5 for details).

Besides the case  $k = 1$ , this result was previously only known when  $\Lambda$  is the space of all polynomials or rational maps of degree  $d$  with marked critical points and  $k$  is maximal [DF, BE, BG].

There is also an “absolute” bifurcation current  $T_{\text{bif}}$ , which was introduced, prior to [DF], by DeMarco [DeM]. When all critical points are marked,  $T_{\text{bif}}$  is just the sum of the associated bifurcation currents. It was shown by Bassanelli and Berteloot that  $\text{Supp}(T_{\text{bif}}^k)$  is contained in the closure of the set of parameters possessing  $k$  distinct neutral (resp. attracting or superattracting) orbits (see [BB1, BB2, Be]). It follows from Corollary 0.2 that  $\text{Supp}(T_{\text{bif}}^k)$  is the closure of the set of parameters where at least  $k$  critical points fall transversely onto repelling cycles.

The arguments required for the proof of Theorem 0.1, along with some classical techniques from value distribution theory also lead to the following interesting characterization of  $T_1 \wedge \dots \wedge T_k$ , in the spirit of higher dimensional holomorphic dynamics.

**Theorem 0.3.** *Let  $(f_\lambda)_{\lambda \in \Lambda}$  be a holomorphic family of rational maps of degree  $d \geq 2$  on  $\mathbb{P}^1$ . Assume that  $c_1, \dots, c_k$  are marked critical points and let  $T_1, \dots, T_k$  be the respective bifurcation currents.*

*There exists a pluripolar set  $\mathcal{E} \subset (\mathbb{P}^1)^k$  such that if  $(z_1, \dots, z_k) \in \mathbb{P}^{1^k} \setminus \mathcal{E}$ , the following equidistribution statement holds:*

$$\frac{1}{d^{nk}} [\{f_\lambda^n(c_1(\lambda)) = z_1\} \cap \dots \cap \{f_\lambda^n(c_k(\lambda)) = z_k\}] \rightarrow T_1 \wedge \dots \wedge T_k.$$

The plan of the paper is the following. After some preliminaries in §1, we prove Theorem 0.3 in §2. In contrast with [BE], the transversality assertion in Theorem 0.1 does not follow from dynamical considerations. It relies on a transversality result of independent interest, which is established in §3. The theorem itself is proved in §4. Finally in §5, we specialize to the case where  $\Lambda$  is the space of polynomial or rational maps of degree  $d$ , and show that Corollary 0.2 can be made more precise in this case.

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## 1. PRELIMINARIES ON BIFURCATION CURRENTS

We first briefly recall from [DF] the construction of the bifurcation current associated to a critically marked family, as well as a few necessary estimates.

We let  $(f_\lambda)_{\lambda \in \Lambda}$  be a family of rational mappings of degree  $d \geq 2$ , with a marked critical point  $c : \lambda \mapsto c(\lambda)$ . Let  $\widehat{f}$  be the product mapping defined on  $\Lambda \times \mathbb{P}^1$  by  $\widehat{f}(\lambda, z) = (\lambda, f_\lambda(z))$ . Fix a Fubini-Study form  $\omega$  on  $\mathbb{P}^1$ . We let  $\pi_\Lambda$  and  $\pi_{\mathbb{P}^1}$  be the coordinate projections on  $\Lambda \times \mathbb{P}^1$  and  $\widehat{\omega} = \pi_{\mathbb{P}^1}^* \omega$ .

It is not difficult to show that the sequence  $d^{-n}(\widehat{f}^n)^*\widehat{\omega}$  converges to a current  $\widehat{T}$ . More precisely we have that

$$(1) \quad d^{-n}(\widehat{f}^n)^*\widehat{\omega} = \widehat{\omega} + dd^c g_n \rightarrow \omega + dd^c g_\infty = \widehat{T}, \text{ with } \|g_n - g_\infty\|_{L^\infty} = O(d^{-n})$$

where the constant in the  $O(\cdot)$  is locally uniform on  $\Lambda$ . Let  $\Gamma_c$  be the graph of  $c$  in  $\widehat{\Lambda}$ . The *bifurcation current associated to  $c$*  is by definition  $T = (\pi_\Lambda)_* \left( \widehat{T}|_{\Gamma_c} \right)$ . Notice that  $T$  has continuous potentials: to be specific, a local potential for  $T$  is given by  $\lambda \mapsto \widehat{g}(\lambda, c(\lambda))$ , where  $\widehat{g}$  is a local potential of  $\widehat{T}$ .

If we let  $c_n$  be defined by  $c_n(\lambda) = f_\lambda^n(c(\lambda))$ , then  $d^{-n}c_n^*\omega = (\pi_\Lambda)_* \left( d^{-n}\widehat{\omega}|_{\widehat{f}^n(\Gamma_c)} \right)$ , so  $d^{-n}c_n^*\omega \xrightarrow{n \rightarrow \infty} T$  and the difference between the potentials is  $O(d^{-n})$ , locally uniformly on  $\Lambda$ .

The following result will be useful. It was first obtained in this general form by Gauthier [Ga, Thm 6.1], with a different proof.

**Lemma 1.1.** *Let as above  $((f_\lambda)_{\lambda \in \Lambda}, c)$  be a critically marked family, and  $T$  be the associated bifurcation current. Then  $T \wedge T = 0$ .*

*Proof.* Uniform convergence of the potentials implies that

$$d^{-n}c_n^*\omega \wedge d^{-n}c_n^*\omega = d^{-2n}c_n^*(\omega^2) \xrightarrow{n \rightarrow \infty} T^2.$$

But  $\omega^2 = 0$  so the result follows.  $\square$

## 2. VALUE DISTRIBUTION OF POST-CRITICAL POINTS

This section is dedicated to Theorem 0.3. Its proof mimicks that of the equidistribution of preimages for endomorphisms of  $\mathbb{P}^k$  so we follow Sibony [S, Theorem 3.6.1], with some extra care due to the fact that  $\Lambda$  is not compact. For clarity we use the formalism of intrinsic capacities of Guedj and Zeriahi [GZ]. Another option would be to adapt the arguments of [DS2].

Before starting the proof, a few remarks on the definitions of pullback currents and measures are in order. It is difficult (if not impossible) to give a reasonable definition of the pullback of a general positive current of bidegree  $(p, p)$  under a holomorphic map. The main trouble is the behavior of pullback currents under weak limits. On the other hand, there are favorable situations where the definition works well:

- when  $T$  is a positive closed current of bidegree  $(1, 1)$ , in which case one can pullback the potentials;
- when the holomorphic map is a submersion.

In our setting, we will consider a sequence of dominant holomorphic mappings  $C_n : \Lambda \rightarrow (\mathbb{P}^1)^k$ , and need to pullback probability measures (i.e. positive currents of bidegree  $(k, k)$  on  $(\mathbb{P}^1)^k$ ). We will consider probability measures  $\nu$  which are products of positive closed currents with bounded potentials  $\nu = S^k$ . Thus,  $\nu$  gives no mass to the set  $\text{CV}(C_n)$  of critical values of  $C_n$ . Likewise,  $(C_n^*S)^k$  is a product of currents with bounded local potentials so it is of locally bounded mass near  $C_n^{-1}(\text{CV}(C_n))$  and gives no mass to it. Therefore, we can define  $C_n^*\nu$  to be the extension of  $C_n^* \left( \nu|_{(\mathbb{P}^1)^k \setminus \text{CV}(C_n)} \right)$  by zero on  $C_n^{-1}(\text{CV}(C_n))$ , and we have that  $C_n^*\nu = (C_n^*S)^k$ .

Pulling back  $\nu$  under  $C_n$  is also compatible with the disintegration  $\nu = \int \delta_a d\nu(a)$ , that is,  $C_n^* \nu = \int C_n^* \delta_a d\nu(a)$ , since again it is enough to restrict to the values of  $a$  lying outside  $\text{CV}(C_n)$ .

Notice that we will never have to consider weak limits on the target space  $(\mathbb{P}^1)^k$ .

*Proof of Theorem 0.3.* By assumption,  $(f_\lambda)_{\lambda \in \Lambda}$  is a holomorphic family of rational mappings of degree  $d \geq 2$  with  $k$  marked critical points  $c_1, \dots, c_k$ . It is no loss of generality to assume that  $\Lambda$  is a ball in  $\mathbb{C}^{\dim(\Lambda)}$ . We let  $C_n : \Lambda \rightarrow (\mathbb{P}^1)^k$  be defined by

$$C_n(\lambda) = (f_\lambda^n(c_1(\lambda)), \dots, f_\lambda^n(c_k(\lambda))) = (c_{n,1}(\lambda), \dots, c_{n,k}(\lambda)).$$

On  $(\mathbb{P}^1)^k$  consider the Kähler form  $\kappa = (k!)^{-1/k}(p_1^* \omega + \dots + p_k^* \omega)$ , where  $p_j = (\mathbb{P}^1)^k \rightarrow \mathbb{P}^1$  is the projection on the  $j^{\text{th}}$  factor. Then  $p_1^* \omega \wedge \dots \wedge p_k^* \omega = \kappa^k$ , and uniform convergence of the potentials implies that  $C_n^*(\kappa^k)$  converges to  $T_1 \wedge \dots \wedge T_k$  as  $n \rightarrow \infty$ .

Our purpose is to show that the set of  $a \in (\mathbb{P}^1)^k$  such that  $d^{-nk} C_n^* \delta_a$  does not converge to  $T_1 \wedge \dots \wedge T_k$  is pluripolar. Fix a test form  $\psi$  of bidegree  $(\dim(\Lambda) - k, \dim(\Lambda) - k)$  on  $\Lambda$ . For fixed  $n$  and  $s > 0$ , we estimate the Monge-Ampère capacity of

$$E_{n,s}^+ := \left\{ a, \frac{1}{d^{nk}} \langle C_n^* \delta_a, \psi \rangle - \frac{1}{d^{nk}} \langle C_n^* \kappa^k, \psi \rangle \geq s \right\}.$$

If  $E_{n,s}^+$  is pluripolar the capacity is zero so there is nothing to prove. This happens in particular when  $C_n$  is not dominant, since in this case  $\langle C_n^* \kappa^k, \psi \rangle = 0$  and  $\langle C_n^* \delta_a, \psi \rangle = 0$  for  $a$  lying outside an analytic set. Hence in what follows we can assume that  $C_n$  is dominant. Consider a non-pluripolar compact subset  $E \subset E_{n,s}^+$  and introduce the so-called Siciak extremal function [GZ, §5.1]

$$v_{E,\kappa} = \sup \{ u \text{ } \kappa\text{-psh, } u \leq 0 \text{ on } E \}.$$

Then the upper semi-continuous regularization  $v := v_{E,\kappa}^*$  is  $\kappa$ -psh, non-negative, and

$$\int_E \kappa_v^k = \int_{(\mathbb{P}^1)^k} \kappa_v^k = \int_{(\mathbb{P}^1)^k} \kappa^k = 1,$$

where  $\kappa_v = \kappa + dd^c v$ . Following [S] we compute

$$\begin{aligned} s &\leq \int_E \left( \frac{1}{d^{nk}} \langle C_n^* \delta_a, \psi \rangle - \frac{1}{d^{nk}} \langle C_n^* \kappa^k, \psi \rangle \right) \kappa_v^k(a) \\ &= \frac{1}{d^{nk}} \int_\Lambda \psi \wedge (C_n^* \kappa_v^k - C_n^* \kappa^k) \\ (2) \quad &= \int_\Lambda \psi \wedge \frac{dd^c(v \circ C_n)}{d^n} \wedge \left( \frac{1}{d^{n(k-1)}} \sum_{j=0}^{k-1} C_n^* \kappa_v^{k-1-j} \wedge C_n^* \kappa^j \right). \end{aligned}$$

Let us estimate the mass (denoted by  $\mathbf{M}(\cdot)$ ) of the current between brackets in the last integral. Let  $u_j$  be a potential of  $c_j^* \omega$  on  $\Lambda$ . Then a potential of  $d^{-n} C_n^* \kappa$  is defined by the formula

$$(3) \quad \frac{1}{(k!)^{1/k}} \sum_{j=1}^k u_j + dd^c(g_{n,j})$$

where  $g_{n,j}$  is as in (1). Observe that this sequence of psh functions is uniformly bounded by some constant  $M$ . Likewise, to obtain a potential of  $d^{-n}C_n^*\kappa_v$ , it is enough to add  $d^{-n}v \circ C_n$  to (3). It then follows from the Chern-Levine-Nirenberg inequality that

$$\begin{aligned} \mathbf{M} \left( \frac{1}{d^{n(k-1)}} \sum_{j=0}^{k-1} C_n^*\kappa_v^{k-1-j} \wedge C_n^*\kappa^j \right) &\leq C^{st} \sum_{j=0}^{k-1} M^j \left( M + \frac{\|v\|_{L^\infty}}{d^n} \right)^{k-1-j} \\ &\leq C^{st} \sum_{j=0}^{k-1} \left( 1 + \frac{\|v\|_{L^\infty}}{d^n} \right)^j. \end{aligned}$$

Plugging this into (2) and integrating by parts we infer that

$$s \leq C^{st} \|\psi\|_{C^2} \frac{\|v\|_{L^\infty}}{d^n} \sum_{j=0}^{k-1} \left( 1 + \frac{\|v\|_{L^\infty}}{d^n} \right)^j.$$

Finally, from elementary calculus we conclude that

$$\frac{\|v\|_{L^\infty}}{d^n} \geq \left( \frac{s}{C^{st} \|\psi\|_{C^2}} + 1 \right)^{1/k} - 1 =: h(s), \text{ that is, } \|v\|_{L^\infty} \geq d^n h(s).$$

The Alexander capacity of  $E$  (relative to  $\kappa$ ) is defined by

$$T_\kappa(E) = \exp(-\sup_X v) = \exp(-\|v\|_{L^\infty})$$

(see [GZ, §5.2]). Thus by the above inequality we get that  $T_\kappa(E) \leq \exp(-d^n h(s)/C \|\psi\|_{C^2})$ , and since  $E \subset E_{n,s}$  is an arbitrary compact subset, the same inequality holds for  $T_\kappa(E_{n,s}^+)$ . We do not need to define the Monge-Ampère capacity  $\text{cap}_\kappa$  precisely here, but only recall that it is a subadditive capacity on  $(\mathbb{P}^1)^k$  which vanishes precisely on pluripolar sets and satisfies an inequality of the form  $\text{cap}_\kappa(E) \leq \frac{-A}{\log T_\kappa(E)}$  when  $T_\kappa(E)$  is small [GZ, §7.1]. So we infer that

$$\text{cap}_\kappa(E_{n,s}^+) \leq \frac{A}{d^n h(s)}.$$

Reversing inequalities in the definition of  $E_{n,s}^+$ , we obtain a similar estimate for

$$E_{n,s} := \left\{ a, \left| \frac{1}{d^{nk}} \langle C_n^* \delta_a, \psi \rangle - \frac{1}{d^{nk}} \langle C_n^* \kappa^k, \psi \rangle \right| \geq s \right\}.$$

By subadditivity of  $\text{cap}_\kappa$ , we infer that for every  $s > 0$ ,  $\text{cap}_\kappa \left( \bigcap_{n_0} \bigcup_{n \geq n_0} E_{n,s} \right) = 0$ , so we conclude that for  $a$  outside a pluripolar set,

$$\frac{1}{d^{nk}} \langle C_n^* \delta_a, \psi \rangle \xrightarrow{n \rightarrow \infty} \langle T_1 \wedge \cdots \wedge T_k, \psi \rangle.$$

To complete the proof it is enough to consider a countable dense family of test forms  $\psi$ .  $\square$

Under an additional global assumption on the family  $(f_\lambda)$ , the following result follows directly from Theorem 0.3.

**Corollary 2.1.** *Let  $(f_\lambda)_{\lambda \in \Lambda}$  be an algebraic family of rational maps of degree  $d \geq 2$  on  $\mathbb{P}^1$ . Assume that  $c_1, \dots, c_k$  are marked critical points and let  $T_1, \dots, T_k$  be the respective bifurcation currents. Let  $C_n$  be defined by  $C_n(\lambda) = (f_\lambda^n(c_1(\lambda)), \dots, f_\lambda^n(c_k(\lambda))) = (c_{n,1}(\lambda), \dots, c_{n,k}(\lambda))$ .*

Then if  $\nu$  is a measure on  $(\mathbb{P}^1)^k$  which gives no mass to pluripolar sets, the sequence of currents  $d^{-nk}C_n^*\nu$  converges to  $T_1 \wedge \cdots \wedge T_k$ .

Here, algebraic means that  $\Lambda$  is an open subset of a quasi-projective variety and that  $(f_\lambda)$  depends algebraically on  $\lambda$ . It is unclear whether this algebraicity assumption is really necessary.

*Proof.* For  $\nu$ -a.e.  $a$ , we have that  $d^{-nk}C_n^*\delta_a \rightarrow T_1 \wedge \cdots \wedge T_k$ . The point is to be able to integrate with respect to  $a$ . Passing to a quasi-projective variety containing  $\Lambda$  if necessary, it is no loss of generality to assume that  $\Lambda$  is quasi-projective. By assumption  $C_n : \Lambda \rightarrow (\mathbb{P}^1)^k$  is a sequence of rational mappings. Therefore for fixed  $n$ , the mass (degree) of  $d^{-nk}C_n^*\delta_a$  is independent of  $a$  on some Zariski open subset of  $(\mathbb{P}^1)^k$ . On the other hand, the average value of this mass is equal to that of  $d^{-nk}C_n^*\kappa^k$  which is bounded in  $n$ . The result then follows from the dominated convergence theorem.  $\square$

### 3. A TRANSVERSALITY THEOREM

As a preliminary step for the proof of Theorem 0.1, in this section we discuss the intersection of uniformly laminar currents of bidegree (1,1) in higher dimension. Beyond the fact that the wedge product of uniformly laminar currents admits (as expected) a geometric interpretation, our main purpose is to show that the intersections between the leaves are generically transverse. The two dimensional case was treated in [BLS] (see also [Du2]).

We first recall some basics on the intersection theory of holomorphic chains in an open set  $\Omega \subset \mathbb{C}^d$ . See Chirka [Ch, Chap. 12] for a fuller account on this. Recall that a *holomorphic chain* in  $\Omega$  is a formal combination  $Z = \sum k_i A_i$  of distinct irreducible analytic subsets of  $\Omega$  with integer coefficients (the *components* of  $Z$ ). The number  $k_i$  is the *multiplicity* of the component  $A_i$ . We'll only have to consider positive chains of pure dimension, that is, for which the multiplicities are positive and the components have the same dimension. The multiplicity of a chain at a point is the sum of that of its components; the *support* of a chain is the union of its components. When there is no risk of confusion we sometimes identify a chain and its support.

A family of holomorphic chains  $Z_1, \dots, Z_k$  in  $\Omega$  is said to *intersect properly* at  $p$  (resp. in  $\Omega$ ) if at  $p$  (resp. for every  $p \in \Omega$ ) the intersection of the supports is of minimal possible dimension, that is,  $\text{codim}_p(\bigcap_{i=1}^k \text{Supp}(Z_i)) = \sum_{i=1}^k \text{codim}_p(\text{Supp}(Z_i))$ . If the intersection of  $Z_1, \dots, Z_k$  is proper at  $p$ , there is a well defined intersection index  $i_p(Z_1, \dots, Z_k)$ , which is constant on the regular part of every irreducible component of  $\text{Supp}(Z_1 \cap \cdots \cap Z_k)$ . If this intersection is proper in  $\Omega$ , then we may define the intersection chain  $Z_1 \wedge \cdots \wedge Z_k$ , whose multiplicity along a given irreducible component of  $\text{Supp}(Z_1 \cap \cdots \cap Z_k)$  is the generic value of  $i_p(Z_1, \dots, Z_k)$ .

If this intersection is not proper, by definition we put  $Z_1 \wedge \cdots \wedge Z_k = 0$ .

If  $Z_1, \dots, Z_k$  are smooth and intersect properly at a given point  $p$ , then the intersection is transverse at  $p$  if and only if  $i_p(Z_1, \dots, Z_k)$  is the product of the multiplicities of the  $Z_i$  at  $p$ . In particular if  $A_1, \dots, A_k$  are submanifolds of  $M$ , and  $C$  is a component of  $A_1 \wedge \cdots \wedge A_k$ , then  $C$  is of multiplicity 1 if and only if the  $A_i$  intersect transversely along the regular part of  $C$ .

Recall that a current  $A$  of bidegree (1,1) in  $\Omega \subset \mathbb{C}^d$  is said to be *uniformly laminar* if there exists a lamination by hypersurfaces in  $\Omega$  such that the restriction to any flow box is of the form  $\int [A_\alpha] da(\alpha)$ , where  $a$  is a positive measure on a transversal and the  $A_\alpha$  are the plaques.

Viewing  $A_\alpha$  as a graph over  $\mathbb{D}^{d-1}$  in  $\mathbb{D}^d$ , we can write  $A_\alpha = \{(z, w), w = h_\alpha(z)\}$ , where  $h_\alpha$  is a bounded holomorphic function on  $\mathbb{D}^{d-1}$ .

Putting  $u_\alpha(z, w) = \log |w - h_\alpha(z)|$ , we get that  $[A_\alpha] = dd^c u_\alpha$ , and the family of psh functions  $u_\alpha$  is locally uniformly bounded in  $L^1_{\text{loc}}$ . Therefore  $u = \int u_\alpha da(\alpha)$  is a well-defined psh function and  $A = dd^c u$ .

Our main result in this section is the following. The result is local so we can work with uniformly laminar currents restricted to flow boxes.

**Theorem 3.1.** *For  $i = 1, \dots, k$ , let  $A_i = \int [A_{i,\alpha_i}] da_i(\alpha_i)$  be uniformly laminar currents with bounded potentials in some open set  $\Omega \subset \mathbb{C}^d$ , and let  $S$  be an irreducible analytic subset of  $\Omega$ . Then the intersection  $A_1 \wedge \dots \wedge A_k \wedge [S]$  is geometric in the sense that*

$$(4) \quad A_1 \wedge \dots \wedge A_k \wedge [S] = \int [A_{1,\alpha_1} \wedge \dots \wedge A_{k,\alpha_k} \wedge S] da_1(\alpha_1) \dots da_k(\alpha_k).$$

*In particular only proper intersections account for the wedge product  $A_1 \wedge \dots \wedge A_k \wedge [S]$ . Furthermore for a.e.  $(\alpha_1, \dots, \alpha_k)$ , the intersection  $A_{1,\alpha_1} \wedge \dots \wedge A_{k,\alpha_k} \wedge S$ , if non-trivial, is a chain of multiplicity 1. In particular the submanifolds  $A_{1,\alpha_1}, \dots, A_{k,\alpha_k}, S$  are transverse along the regular part of their intersection.*

*Proof.* Since  $A_1 \wedge \dots \wedge A_k \wedge [S]$  carries no mass on  $\text{Sing}(S)$  we may suppose that  $S$  is smooth. We argue by induction on  $k$ . Assume that for some  $0 \leq q < k$  the result holds for  $A_1 \wedge \dots \wedge A_q \wedge [S]$  (of course of  $q = 0$  this expression simply means  $[S]$ , and the result is true in this case). Fix  $\alpha_1, \dots, \alpha_q$  and let  $V = \text{Supp}(A_{1,\alpha_1} \wedge \dots \wedge A_{q,\alpha_q} \wedge S)$ . Write  $A_{q+1,\alpha_{q+1}} = \{w = h_{q+1,\alpha_{q+1}}(z)\}$ , with  $|h_{q+1,\alpha_{q+1}}| < 1$ , in some local system of coordinates. As above, the function  $u_{q+1} := \int \log |w - h_{q,\alpha_q}(z)| da_q(\alpha_q)$  is a potential of  $A_q$ , hence bounded by assumption, therefore  $u_{q+1} := \int \log |h_{q+1,\alpha_{q+1}}| da_{q+1}(\alpha_{q+1})$  is locally integrable on  $V$ . Since the logarithms are negative, by Fubini's theorem we infer that for a.e.  $\alpha_{q+1}$ ,  $\log |h_{q+1,\alpha_{q+1}}| \in L^1_{\text{loc}}(V)$ , so  $V \not\subset A_{q+1,\alpha_{q+1}}$ . In particular for a.e.  $\alpha_{q+1}$ , the intersection  $V \cap A_{q+1,\alpha_{q+1}}$  is proper.

The justification of the formula (4) is classical. We rely on the following two facts:

- If  $u = \int u_\alpha da(\alpha)$  is an integral of negative psh functions and  $T$  is a positive current such that  $uT$  (resp  $u_\alpha T$  for a.e.  $\alpha$ ) has finite mass, then  $uT = \int u_\alpha T da(\alpha)$ .
- Likewise if  $T = \int T_\alpha da(\alpha)$  is an integral of positive currents and  $u$  is a negative psh function such that  $uT$  (resp.  $uT_\alpha$  for a.e.  $\alpha$ ) has finite mass, then  $uT = \int uT_\alpha da(\alpha)$ .

Assume by induction that (4) holds for  $k = q$ , and write

$$\begin{aligned} (5) \quad A_1 \wedge \dots \wedge A_q \wedge A_{q+1} \wedge [S] &= dd^c (u_{q+1} A_1 \wedge \dots \wedge A_q \wedge [S]) \\ &= dd^c \left( u_{q+1} \int [A_{1,\alpha_1} \wedge \dots \wedge A_{q,\alpha_q} \wedge S] da_1(\alpha_1) \dots da_q(\alpha_q) \right) \\ &= dd^c \left( \int u_{q+1} [A_{1,\alpha_1} \wedge \dots \wedge A_{q,\alpha_q} \wedge S] da_1(\alpha_1) \dots da_q(\alpha_q) \right) \\ &= \int ((dd^c u_{q+1}) \wedge [A_{1,\alpha_1} \wedge \dots \wedge A_{q,\alpha_q} \wedge S]) da_1(\alpha_1) \dots da_q(\alpha_q). \end{aligned}$$

Now, as before, fix  $\alpha_1, \dots, \alpha_q$  and put  $V = A_{1,\alpha_1} \wedge \dots \wedge A_{q,\alpha_q} \wedge S$  (here  $V$  is the chain, not the mere support), and consider

$$u_{q+1}[V] = \left( \int \log |h_{q+1,\alpha_{q+1}}| da_{q+1}(\alpha_{q+1}) \right) [V].$$

Let  $\mathcal{A}_R$  be the set of values  $\alpha_{q+1}$  so that  $\|\log |h_{q+1, \alpha_{q+1}}|\|_{L^1(V)} \leq R$ , and let

$$u_{q+1, R} = \int_{\mathcal{A}_R} \log |h_{q+1, \alpha_{q+1}}| da_{q+1}(\alpha_{q+1}).$$

Notice that  $u_{q+1, R}$  decreases to  $u_{q+1}$  as  $R$  increases to infinity. We then infer that

$$u_{q+1, R}[V] = \int_{\mathcal{A}_R} \log |h_{q+1, \alpha_{q+1}}| [V] da_{q+1}(\alpha_{q+1}).$$

Thus by monotone convergence we obtain that  $u_{q+1}[V] = \int \log |h_{q+1, \alpha_{q+1}}| [V] da_{q+1}(\alpha_{q+1})$ . Finally, by taking the  $dd^c$ , we conclude that

$$(dd^c u_{q+1}) \wedge [V] = \int [A_{q+1, \alpha_{q+1}}] \wedge [V] da_{q+1}(\alpha_{q+1}) = \int [A_{q+1, \alpha_{q+1}} \wedge V] da_{q+1}(\alpha_{q+1}),$$

where the second equality comes from the fact that intersection in the sense of currents of properly intersecting analytic sets coincides with intersection in the sense of chains [Ch, §16.2]. This, together with (5), concludes the proof of (4).

We now show that intersections are generically transverse. Again we argue by induction, so let us assume that for  $q < k$  the result holds for  $A_1 \wedge \cdots \wedge A_q \wedge [S]$ . Then for a.e.  $\alpha_1, \dots, \alpha_q$ , every component of  $A_{1, \alpha_1} \wedge \cdots \wedge A_{q, \alpha_q} \wedge S$  is of multiplicity 1. Since  $A_{q+1}$  has bounded potential, the intersection  $[A_{1, \alpha_1} \wedge \cdots \wedge A_{q, \alpha_q} \wedge S] \wedge A_{q+1}$  gives no mass to the singular locus of  $A_{1, \alpha_1} \wedge \cdots \wedge A_{q, \alpha_q} \wedge S$ . Therefore in the geometric intersection

$$A_1 \wedge \cdots \wedge A_{q+1} \wedge [S] = \int [A_{1, \alpha_1} \wedge \cdots \wedge A_{q, \alpha_q} \wedge A_{q+1, \alpha_{q+1}} \wedge S] da_1(\alpha_1) \cdots da_{q+1}(\alpha_{q+1})$$

we can restrict to those intersections lying in the regular part of  $A_{1, \alpha_1} \wedge \cdots \wedge A_{q, \alpha_q} \wedge S$ .

So let us consider a proper intersection between a smooth part of  $A_{1, \alpha_1} \wedge \cdots \wedge A_{q, \alpha_q} \wedge S$  (which will be fixed from now on) and some leaf  $A_{q+1, \alpha_{q+1}}$  of  $A_{q+1}$ . Let  $W$  be a component of this intersection (hence of codimension  $q+1 + \text{codim}(S)$ ). We want to show that for generic  $\alpha_{q+1}$ ,  $W$  has multiplicity 1. Let  $p$  belonging to the regular part of  $W$ . Changing local coordinates, it is no loss of generality to assume that in the neighborhood of  $p$ ,  $A_{1, \alpha_1} \wedge \cdots \wedge A_{q, \alpha_q} \wedge S$  is an affine subspace of codimension  $q + \text{codim}(S)$ . If  $A_{1, \alpha_1} \wedge \cdots \wedge A_{q, \alpha_q} \wedge S$  and  $A_{q+1, \alpha_{q+1}}$  are transverse at  $p$  there is nothing to prove. Otherwise  $A_{1, \alpha_1} \wedge \cdots \wedge A_{q, \alpha_q} \wedge S$  is contained in the tangent space  $T_p(A_{q+1, \alpha_{q+1}})$ . Let  $\Pi$  be the 2-dimensional linear subspace through  $p$ , generated by  $(e, f)$ , where  $e$  is a vector transverse to  $W$  in  $A_{1, \alpha_1} \wedge \cdots \wedge A_{q, \alpha_q} \wedge S$ , and  $f$  is transverse to  $T_p(A_{q+1, \alpha_{q+1}})$ . Then  $\Pi \cap A_{1, \alpha_1} \wedge \cdots \wedge A_{q, \alpha_q} \wedge S$  is a smooth curve: in our coordinates it is the line  $L$  tangent to  $e$  through  $p$ . Note also that  $W \cap \Pi = \{p\}$ . By transversality,  $A_{q+1}$  induces a lamination by curves on  $\Pi$  near  $p$ , which is tangent to  $e$  at  $\Pi$ , and the leaf through  $p$  has an isolated intersection with  $L$ . In this 2-dimensional situation, [BLS, Lemma 6.4] implies that for *every*  $\alpha$  close to  $\alpha_{q+1}$ ,  $\Pi \cap A_{q+1, \alpha}$  intersects  $L$  transversely in  $\Pi$ , hence  $A_{1, \alpha_1} \wedge \cdots \wedge A_{q, \alpha_q} \wedge S$  is transverse to  $A_{q+1, \alpha}$  in  $\Omega$ . This finishes the proof.  $\square$

*Remark 3.2.* The argument was written so that the proof of (4), as well as generic properness of intersections, carries over without modification to the case where the  $A_i$  are *uniformly woven* currents, that is, when the leaves are allowed to intersect. On the other hand generic transversality needn't hold in this more general context: it already fails in dimension 2, as simple examples show.



## 4. FALLING ONTO REPELLING CYCLES

In this section we prove Theorem 0.1. Near  $\lambda_0 \in \text{Supp}(T_1 \wedge \cdots \wedge T_k)$ , we want to construct parameters at which the marked critical points fall onto repelling cycles. The strategy is to first make these critical points fall into a (persistent) non-polar hyperbolic set, and then use the transversality results of the previous section to “push” them towards repelling cycles.

*Proof of Theorem 0.1.* The first step is the following classical lemma.

**Lemma 4.1.** *Let  $f$  be a rational map of degree  $d \geq 2$  on  $\mathbb{P}^1$ . There exists an integer  $m$  and a  $f^m$ -invariant compact set  $K$  such that:*

- $f^m|_K$  is uniformly hyperbolic and conjugated to a one-sided shift on two symbols;
- the unique balanced measure  $\nu$  on  $K$  (that is, such that  $(f^m)^*\nu = 2\nu$ ) has (Hölder) continuous potential.

*Proof.* It follows from the proof of the equidistribution of repelling points (see [L]) that there exists an open ball  $B$ , an integer  $m$  and two univalent inverse branches  $f_1^{-m}$  and  $f_2^{-m}$  of  $f^m$  on  $B$  such that  $f_i^{-m}(B) \Subset B$  and  $\overline{f_1^{-m}(B)} \cap \overline{f_2^{-m}(B)} = \emptyset$ . Thus, letting  $B_i = f_i^{-m}(B)$ , we may consider

$$K = \{z \in B_1 \cup B_2, \forall n \in \mathbb{N}, f^{mn}(z) \in B\}.$$

It is well-known that  $f^m|_K$  is uniformly hyperbolic and conjugate to a one-sided 2-shift.

In addition,  $K$  is not polar. More precisely it can be shown that the unique balanced measure under  $f^n$  has Hölder continuous potential, see [DS1, Theorem 3.7.1] for a proof.  $\square$

Let  $(f_\lambda)$  be a family of rational mappings with  $k$  marked critical points as in the statement of Theorem 0.1, and let  $\lambda_0$  be a parameter in  $\text{Supp}(T_1 \wedge \cdots \wedge T_k)$ . We apply Lemma 4.1 to  $f_{\lambda_0}$ , which provides us with a  $f_{\lambda_0}^m$ -invariant hyperbolic set  $K_{\lambda_0}$ , endowed with a measure  $\nu_{\lambda_0}$ . Replacing the family  $(f_\lambda)$  by  $(f_\lambda^m)$ , it is no loss of generality to assume that  $m = 1$ . Note that  $c_1, \dots, c_k$  are still marked critical points, and the associated bifurcation currents are the same. Also, we rename  $\lambda_0$  into 0.

The hyperbolic set  $K_0$  persists in some neighborhood of 0. More precisely there exists a neighborhood  $N$  of 0, biholomorphic to a ball, and a holomorphic motion  $(h_\lambda)_{\lambda \in N}$  of  $K_{\lambda_0}$  such that  $h_\lambda \circ f_0 = f_\lambda \circ h_\lambda$ . We set  $K_\lambda = h_\lambda(K_0)$ , and  $\nu_\lambda = (h_\lambda)_*\nu_0$ . Without loss of generality we rename  $N$  into  $\Lambda$ .

In  $\Lambda \times \mathbb{P}^1$ , consider the uniformly laminar current induced by the holomorphic motion of the measure  $\nu_0$ . More precisely, for  $z \in K_0$ , we let  $\widehat{h} \cdot z$  be the graph of the holomorphic motion over  $\Lambda$  passing through  $z$ , and let  $\widehat{\nu}$  be the (1,1) positive closed current defined by

$$(6) \quad \widehat{\nu} = \int [\widehat{h} \cdot z] d\nu_0(z).$$

**Lemma 4.2.**  *$\widehat{\nu}$  has bounded potentials.*

*Proof.* Since  $K_0$  is contained in some affine chart  $\mathbb{C} \subset \mathbb{P}^1$ , reducing  $\Lambda$  again if necessary, we may view  $\widehat{\nu}$  as a *horizontal current* in  $\Lambda \times \mathbb{C}$ , in the sense that  $\text{Supp } \widehat{\nu} \cap (\{\lambda\} \times \mathbb{C})$  is compact for every  $\lambda \in \Lambda$ . It is classical that a potential of  $\widehat{\nu}$  is then given by

$$u(\lambda, z) = \int \log |s - z| d\nu_\lambda(s).$$

A proof of this fact is given in dimension 2 in [Du1, §6], which easily adapts to higher dimensions. The Hölder continuity of holomorphic motions implies that, restricted to  $\text{Supp}(\nu_\lambda)$ ,

$u(\lambda, \cdot)$  is Hölder continuous as a function of  $z$ , locally uniformly in  $\lambda$ . In particular  $u$  is locally uniformly bounded on  $\text{Supp}(\widehat{\nu})$ , hence everywhere, by the maximum principle. The result follows. Adapting [Du1, §6], it is not difficult to see that  $u$  is actually locally Hölder continuous.  $\square$

Let us now work in  $\Lambda \times (\mathbb{P}^1)^k$  and, similarly to Theorem 0.3, investigate the motion of the  $k$  marked critical points relative to  $K_\lambda$ . For  $1 \leq j \leq k$  let  $\widehat{\nu}_j = \widehat{p}_j^* \widehat{\nu}$ , where  $\widehat{p}_j : \Lambda \times (\mathbb{P}^1)^k \rightarrow \Lambda \times \mathbb{P}^1$  is defined by  $(\lambda, z_1, \dots, z_k) \mapsto (\lambda, z_j)$ , and  $\widehat{\nu}$  is as in (6). We put  $V = (k!)^{-1/k} (\widehat{\nu}_1 + \dots + \widehat{\nu}_k)$  so that  $V^k = \widehat{\nu}_1 \wedge \dots \wedge \widehat{\nu}_k$ , and letting  $\widehat{C}_n(\lambda) := (\lambda, C_n(\lambda))$ , we study the sequences of currents  $\widehat{C}_n^* V$  and  $\widehat{C}_n^* V^k$ .

Note that we are abusing slightly here since there does not exist a well-defined pull back operator  $\widehat{C}_n^*$  from currents on  $\Lambda \times (\mathbb{P}^1)^k$  to currents on  $\Lambda$ , even for positive closed currents of bidegree  $(1,1)$ , since  $\widehat{C}_n$  is obviously not dominant. The meaning of this notation here is the following:  $V$  is a positive closed current of bidegree  $(1,1)$  on  $\Lambda \times (\mathbb{P}^1)^k$  with bounded potentials, so the wedge product  $V \wedge [\Gamma(C_n)]$  of  $V$  with the graph of  $C_n$  is well defined, and corresponds to the restriction of  $V$  to  $\Gamma(C_n)$ . On the other hand  $\pi_\Lambda|_{\Gamma(C_n)} : \Gamma(C_n) \rightarrow \Lambda$  is a biholomorphism, so we can set  $\widehat{C}_n^* V = (\pi_\Lambda)_* (V \wedge [\Gamma(C_n)])$ . Writing locally  $V = dd^c_{(\lambda,z)}(v(\lambda, z))$ , we obtain that  $\widehat{C}_n^* V = dd^c_\lambda(v(\lambda, C_n(\lambda)))$  –recall that a psh function has a value at every point.

The definition of  $\widehat{C}_n^* V^k$  is similar, arguing by induction, since for every  $1 \leq j \leq k$ ,  $V^j \wedge [\Gamma(C_n)]$  is well-defined. These definitions coincide of course with the usual ones for smooth forms, and are stable under regularizations.

This being said, it is easy that  $d^{-n} \widehat{C}_n^* V$  converges to  $(k!)^{-1/k} (T_1 + \dots + T_k)$ , where in addition the potentials converge uniformly. Hence  $d^{-n} \widehat{C}_n^* V^k$  converges to  $T_1 \wedge \dots \wedge T_k$ .

Indeed, let  $\widehat{\kappa}$  be the pull back of the previously defined Kähler form  $\kappa$  under the natural projection  $\Lambda \times (\mathbb{P}^1)^k \rightarrow (\mathbb{P}^1)^k$ . We know from the proof of Theorem 0.3 that the sequence  $d^{-n} \widehat{C}_n^* \widehat{\kappa} = d^{-n} C_n^* \kappa$  converges to  $(k!)^{-1/k} (T_1 + \dots + T_k)$ , with uniform convergence of the potentials. Now we simply write  $V - \widehat{\kappa} = dd^c w$ , where  $w$  is a uniformly bounded function on  $\Lambda \times (\mathbb{P}^1)^k$ , so

$$d^{-n} \widehat{C}_n^* V = d^{-n} \widehat{C}_n^* \widehat{\kappa} + d^{-n} dd^c(w \circ \widehat{C}_n),$$

and the result follows.

We now give a geometric interpretation of the current  $V^k$  and its intersection with  $\Gamma(C_n)$ . For  $(z_1, \dots, z_k) \in K_0^k$ , we let

$$\widehat{h} \cdot (z_1, \dots, z_k) = \{(\lambda, h_\lambda(z_1), \dots, h_\lambda(z_k)), \lambda \in \Lambda\}$$

be the graph of its continuation under the product holomorphic motion. The first observation is that the current  $V^k$  is an integral of graphs over  $\Lambda$ :

$$(7) \quad V^k = \widehat{\nu}_1 \wedge \dots \wedge \widehat{\nu}_k = \int [\widehat{h} \cdot (z_1, \dots, z_k)] d\nu(z_1) \cdots d\nu(z_k).$$

Indeed, it is clear that for every  $j$ ,  $\widehat{\nu}_j$  is a uniformly laminar current, since it is the pullback under  $\widehat{p}_j$  of the uniformly laminar current  $\widehat{\nu}$ . The geometric interpretation (7) of  $V^k$  then follows from Theorem 3.1. Applying now the same theorem to  $V^k \wedge [\Gamma(C_n)]$ , we obtain that

$$(8) \quad V^k \wedge [\Gamma(C_n)] = \int [\widehat{h} \cdot (z_1, \dots, z_k) \wedge \Gamma(C_n)] d\nu(z_1) \cdots d\nu(z_k),$$

and furthermore<sup>1</sup> that the intersection chain  $\widehat{h} \cdot (z_1, \dots, z_k) \wedge \Gamma(C_n)$  is generically of multiplicity 1. Recall that this means that the intersection  $\widehat{h} \cdot (z_1, \dots, z_k) \cap \Gamma(C_n)$  is geometrically transverse along  $\text{Reg}(\widehat{h} \cdot (z_1, \dots, z_k) \wedge \Gamma(C_n))$ , which is of full trace measure in  $[\widehat{h} \cdot (z_1, \dots, z_k) \wedge \Gamma(C_n)]$ .

We are now in position to conclude the proof. Since  $d^{-n} \widehat{C}_n^* V^k = (\pi_\Lambda)_*(V^k \wedge [\Gamma(C_n)])$  converges to  $T_1 \wedge \dots \wedge T_k$  and  $\lambda_0 = 0 \in \text{Supp}(T_1 \wedge \dots \wedge T_k)$ , then by (8), there exists a sequence of parameters  $\lambda_n$  converging to 0 such that  $\Gamma(C_n)$  and some graph  $\widehat{h} \cdot (z_1^n, \dots, z_k^n)$  of the holomorphic motion intersect transversely over  $\lambda_n$ . Repelling periodic orbits for  $f_0$  are dense in  $K_0$ , so for fixed (large)  $n$ , for  $j = 1, \dots, k$ , there exists a sequence of repelling  $f_0$ -periodic points  $(z_j^{n,q})_{q \geq 1}$  belonging to  $K_0$ , and converging to  $z_j^n$ . The continuations  $h_\lambda(z_j^{n,q})$  remain repelling throughout  $\Lambda$  by hyperbolicity. By the persistence of transverse intersections, for large  $q$ ,  $\widehat{h} \cdot (z_1^{n,q}, \dots, z_k^{n,q})$  intersects  $\Gamma(C_n)$  transversely near  $\lambda_n$ . Thus we have found parameters close to 0 at which  $c_1, \dots, c_k$  fall transversely onto repelling cycles.  $\square$

## 5. WHEN $\Lambda$ IS THE SPACE OF POLYNOMIAL OR RATIONAL MAPS

Here we show that in the case where  $\Lambda$  is the space of all polynomials or rational maps with all critical points marked, the statement of Corollary 0.2 takes a simpler form. Similar ideas already appeared in the polynomial case in [Ga, §8].

We say that a family  $(f_\lambda)$  of rational maps is *reduced* if it is generically transverse to the orbits of the  $\text{PSL}(2, \mathbb{C})$ -action by conjugacy on the space of rational maps of degree  $d$ . In other words, we require that for every  $\lambda_0 \in \Lambda$ , the set of parameters  $\lambda \in \Lambda$  such that  $f_\lambda$  is holomorphically conjugate to  $f_{\lambda_0}$  is discrete.

We start with a general result.

**Proposition 5.1.** *Let  $(f_\lambda)_{\lambda \in \Lambda}$  be a reduced algebraic family of rational maps of degree  $d$ , with marked critical points  $c_1, \dots, c_\ell$ . Assume that for all  $2\ell$ -tuples of integers  $(n_j \geq 0, m_j \geq 1)_{1 \leq j \leq \ell}$ , the subvariety defined by  $\bigcap_{j=1}^\ell \left\{ \lambda, f_\lambda^{n_j}(c_j(\lambda)) = f_\lambda^{n_j+m_j}(c_j(\lambda)) \right\}$ , whenever non-empty, is of pure codimension  $\ell$ .*

*Then for every  $k \leq \ell$ ,*

$$\text{Supp}(T_1 \wedge \dots \wedge T_k) = \overline{\{\lambda, c_1(\lambda), \dots, c_k(\lambda) \text{ fall onto repelling cycles}\}}$$

We use a result of Gauthier [Ga, Thm 6.2], that we briefly describe now. Assume that for some  $\lambda_0 \in \Lambda$ , the marked critical points  $c_j$  eventually land on repelling periodic points  $p_j$ , that is  $f_{\lambda_0}^{n_j}(c_j(\lambda_0)) = p_j(\lambda_0)$ . As in the introduction, let  $C$  and  $P$  be respectively defined by  $C : \lambda \mapsto (f_\lambda^{n_1}(c_1(\lambda)), \dots, f_\lambda^{n_k}(c_k(\lambda)))$  and  $P : \lambda \mapsto (p_1(\lambda), \dots, p_k(\lambda))$ , where  $p_j(\lambda)$  is the natural continuation of  $p_j(\lambda_0)$ . We say that the critical points  $c_j$  *fall properly* onto the respective repelling points  $p_j$  at  $\lambda_0$  if the graphs of the two mappings  $\lambda \mapsto (p_1(\lambda), \dots, p_k(\lambda))$  and  $\lambda \mapsto (f_\lambda^{n_1}(c_1(\lambda)), \dots, f_\lambda^{n_k}(c_k(\lambda)))$  intersect properly at  $(\lambda_0, p_1(\lambda_0), \dots, p_k(\lambda_0))$ . Denoting by  $m_j$  the period of  $p_j$ , we see that for this it is enough that the subvariety  $\bigcap_{j=1}^k \left\{ \lambda, f_\lambda^{n_j}(c_j(\lambda)) = f_\lambda^{n_j+m_j}(c_j(\lambda)) \right\}$  has codimension  $k$  at  $\lambda_0$ . Gauthier's theorem asserts that if the critical points  $c_j$  fall properly onto the respective repelling points  $p_j$  at  $\lambda_0$ , then  $\lambda_0 \in \text{Supp}(T_1 \wedge \dots \wedge T_k)$ .

<sup>1</sup>Notice that while the transversality of the leaves of the  $\widehat{v}_j$  is obvious, this is not anymore the case after restriction to  $\Gamma(C_n)$ . This is where we need the full strength of Theorem 3.1.

*Proof.* In view of Theorem 0.1, we only need to show that if the  $c_j(\lambda_0)$ ,  $1 \leq j \leq k$  eventually land on repelling periodic points, then  $\lambda_0 \in \text{Supp}(T_1 \wedge \cdots \wedge T_k)$ . For this, in order to use the above result, we show that under the assumptions of the proposition, for every  $k \leq \ell$ , and all integers  $(n_j \geq 0, m_j \geq 1)_{1 \leq j \leq k}$ ,  $\bigcap_{j=1}^k \left\{ \lambda, f_\lambda^{n_j}(c_j(\lambda)) = f_\lambda^{n_j+m_j}(c_j(\lambda)) \right\}$ , whenever non-empty, is of pure codimension  $k$  (notice that this does not follow directly from the assumption of the proposition, see [Ch, pp. 143-144]). The proof is by decreasing induction on  $k$ , so assume that the result holds at step  $k+1$ , and let  $W$  be an irreducible component of  $\bigcap_{j=1}^k \left\{ \lambda, f_\lambda^{n_j}(c_j(\lambda)) = f_\lambda^{n_j+m_j}(c_j(\lambda)) \right\}$  for arbitrary integers  $(n_j \geq 0, m_j \geq 1)_{1 \leq j \leq k}$ . A first possibility is that  $c_{k+1}$  is passive on  $W$ . Since  $W$  is algebraic, [DF, Thm 2.5] asserts that either all mappings on  $W$  are holomorphically conjugate, or  $c_{k+1}$  is persistently preperiodic on  $W$ . The first case is excluded because  $(f_\lambda)$  is reduced, and, since  $\text{codim}(W) \leq k$ , the second case contradicts the induction hypothesis. We then infer that the activity locus of  $c_{k+1}$  along  $W$  is non-empty. Therefore, there are non-empty proper hypersurfaces in  $W$  where  $c_{k+1}$  becomes preperiodic. By the induction hypothesis the codimension of such a hypersurface in  $\Lambda$  equals  $k+1$ , hence  $\text{codim}(W) = k$ .  $\square$

Let  $\mathcal{P}_d^{cm}$  be the space of polynomials of degree  $d$  with marked critical points, up to affine conjugacy. This space is described in detail in [DF]: it is an affine algebraic variety of dimension  $d-1$ , which admits a finite branched cover by  $\mathbb{C}^{d-1}$ . Denote by  $(c_j)_{j=1,\dots,d-1}$  the marked critical points and by  $T_j$  the associated bifurcation currents. Applying Proposition 5.1 we thus recover the following result from [Ga, §8].

**Corollary 5.2.** *In  $\mathcal{P}_d^{cm}$ , for every  $k \leq d-1$ , we have that*

$$\text{Supp}(T_1 \wedge \cdots \wedge T_k) = \overline{\{\lambda, c_1(\lambda), \dots, c_k(\lambda) \text{ fall onto repelling cycles}\}}$$

*Proof.* The assumption of Proposition 5.1 is satisfied for  $\ell = d-1$ . Indeed, for all  $(n_j \geq 0, m_j \geq 1)_{1 \leq j \leq d-1}$ ,  $\bigcap_{j=1}^{d-1} \left\{ \lambda, f_\lambda^{n_j}(c_j(\lambda)) = f_\lambda^{n_j+m_j}(c_j(\lambda)) \right\}$  is of dimension 0, because it is contained in the connectedness locus, which is compact in  $\mathbb{C}^{d-1}$  by Branner-Hubbard [BH].  $\square$

Let now  $\mathcal{M}_d^{cm}$  be the space of rational mappings of degree  $d$ , up to Möbius conjugacy, with marked critical points  $(c_j)_{j=1,\dots,2d-2}$ . It is a normal quasiprojective variety of dimension  $2d-2$  [Sil]. Proposition 5.1 then leads to the following:

**Corollary 5.3.** *In  $\mathcal{M}_d^{cm}$ , for every  $k \leq 2d-2$ , we have that*

$$\text{Supp}(T_1 \wedge \cdots \wedge T_k) = \overline{\{\lambda, c_1(\lambda), \dots, c_k(\lambda) \text{ fall onto repelling cycles}\}}$$

*Proof.* A difficulty here is that the assumption of proposition 5.1 is not valid in general for  $\ell = 2d-2$  due to the possibility of *flexible Lattès examples* (see Milnor [Mi] for a general account on Lattès examples). So we treat the cases  $k \leq 2d-3$  and  $k = 2d-2$  separately. For  $k = 2d-2$ , the result was proven by Buff and Gauthier [BG].

Let us show that the assumption of Proposition 5.1 holds for  $k = 2d-3$ . For this, assume that for some  $(n_j, m_j)_{1 \leq j \leq 2d-1}$  as above,  $\bigcap_{j=1}^{2d-3} \left\{ \lambda, f_\lambda^{n_j}(c_j(\lambda)) = f_\lambda^{n_j+m_j}(c_j(\lambda)) \right\}$  is not empty, and let us prove that it is of pure codimension  $2d-3$  (i.e. of pure dimension 1). If not, it admits a component  $W$  of dimension greater than 1. If the last free critical point  $c_{2d-2}$  is passive on  $W$ , then the family  $(f_\lambda)$  is stable along  $W$ . It then follows from a theorem of McMullen [McM] that  $W$  is a (reduced) family of flexible Lattès examples, which is impossible because such a

family should have dimension 1. Therefore, the activity locus of  $c_{2d-2}$  is non-empty, giving rise to algebraic hypersurfaces  $H_m \subset W$  such that where  $c_{2d-2}$  is persistently preperiodic to periodic points of arbitrary large period  $m$  (of course the minimal period may drop to some divisor of  $m$ , but this only happens on a proper subvariety of  $H_m$ ). As before,  $H_m$  must be a family of flexible Lattès examples, which is impossible for in this case the critical points must eventually fall on repelling points of period 1 or 2 (see [Mi]). This contradiction finishes the proof.  $\square$

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